

The Unitary Transformation in Quantum Teleportation

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February 1, 2008

Abstract

In the well known treatment of quantum teleportation, the receiver should convert the state of his EPR particle into the replica of the unknown quantum state by one of four possible unitary transformations. However, the importance of these unitary transformations must be emphasized. We will show in this paper that the receiver can not transform the state of his particle into an exact replica of the unknown state which the sender want to transfer if he have not a proper implementation of these unitary transformations. In the procedure of converting state, the inevitable coupling between EPR particle and environment which is needed by the implementation of unitary transformations will reduce the accuracy of the replica.

03.67.Hk, 03.67.-a, 03.65.Ta..

In 1993, Bennett et al.[1] proposed a famous treatment to transfer an intact quantum state from one place to another by use of the long-range correlation between EPR pair of particles. In their scheme, an unknown state and one of EPR particles are given to the sender, and the sender then perform a complete measurement on the joint system of the unknown state and her EPR state. After this, the receiver will perform a unitary transformation on the second particle of the EPR pair to obtain the replica of the unknown state, certainly, this unitary transformation is determined by the results of measurement told by the sender through a classical channel. The experimental realizations of their treatment were exhibited by Bouwmeester et al.[2] and Boschi et al.[3], respectively, in which an initial photon which carries the polarization is transferred by use of a pair of entangled photons prepared in an EPR state. These approaches to quantum teleportation had inspired many investigations into this field, such as the discussions of continuous variable quantum teleportation[4, 5, 6], the analysis of quantum fluctuation in the teleportation[7] etc.

However, it should be pointed out that the physical implementation of the unitary transformations on the second EPR particle should be noticed, they

are not mere the pure mathematical transformations, we need realize these unitary transformations by other physical systems. For convenience, we summarily call these system 'environment'. The inevitable interaction between EPR particle in the hands of receiver and the environment will affect the replica of the unknown state. Even worse, if we have not properly chosen the physical realization of these unitary transformations, the unknown quantum state will not be teleported enough accurately because of the influence of environment.

Suppose a sender, traditionally called 'Alice', who wish to communicate an unknown quantum state $|\Psi\rangle = a|0\rangle + b|1\rangle$ of spin-1/2 particle (particle 1) to a receiver, 'Bob'. Two other spin-1/2 particles are prepared in an EPR singlet state. According to Bennett et al.'s treatment, one EPR particle (particle 2) is given to Alice, the other (particle 3) is given to Bob. Alice makes a combined measurement on her EPR particle 2 and the unknown particle 1, then Bob's particle 3 will be in one of the following four pure states: $-a|0\rangle_3 - b|1\rangle_3$, $-a|0\rangle_3 + b|1\rangle_3$, $b|0\rangle_3 + a|1\rangle_3$, and $-b|0\rangle_3 + a|1\rangle_3$. In the ideal case, Bob can convert the state of particle 3 into an exact replica of the initial state $|\Psi\rangle = a|0\rangle + b|1\rangle$ by a unitary transformation which depends on the results of measurement told by Alice via classical channel. However, these unitary transformations must be performed through other physical systems or apparatus, which can be summarily described as 'environment', then the interaction between particle 3 and the 'environment' occurs, which will couple the quantum state of environment with particle 3 and violate the accurate replica of the initial quantum state $|\Psi\rangle$.

In fact, the above interaction between particle 3 and environment makes the quantum state of the combined system (particle 3-environment) evolves as follows[8]

$$\begin{aligned} |\Phi(t = t_0)\rangle &= |E_0\rangle \otimes |\Psi\rangle_3 \\ \longrightarrow |\Phi(t > t_1)\rangle &= C_0 a |E_0\rangle |0\rangle_3 + C_1 b |E_1\rangle |1\rangle_3. \end{aligned} \quad (1)$$

In the above, $|E_0\rangle$, $|E_1\rangle$ are the state vectors of environment, while $|\Psi\rangle_3$ describes the quantum state of particle 3. As a result of particle 3-environment interaction, the correlation between particle and environment has been established after time t_1 , the state vectors of particle 3 and environment have coupled to each other after time t_1 . Expression (1) clearly demonstrates the violation of pure state $|\Psi\rangle_3$ after the practical implementation of unitary transformations. We can also show this violation by its density matrix. The reduced density matrix of particle 3 is

$$\begin{aligned} \rho_3 &= Tr_E[|\Phi(t > t_1)\rangle\langle\Phi(t > t_1)|] \\ &= (|C_0 a|^2 + |C_0 a|^2 |\langle E_1 | E_0 \rangle|^2) |0\rangle\langle 0| + (2C_0 C_1^* a b^* \langle E_1 | E_0 \rangle) |0\rangle\langle 1| \\ &\quad + (2C_1 C_0^* b a^* \langle E_0 | E_1 \rangle) |1\rangle\langle 0| + (|C_1 b|^2 + |C_1 b|^2 |\langle E_0 | E_1 \rangle|^2) |1\rangle\langle 1|. \end{aligned} \quad (2)$$

When the state vectors $|E_0\rangle$, $|E_1\rangle$ of environment are orthogonal to each other,

the density matrix can reduce to

$$\rho_3 = |C_0 a|^2 |0\rangle\langle 0| + |C_1 b|^2 |1\rangle\langle 1|, \quad (3)$$

which indicates pure state $|\Psi\rangle_3$ has become to a mixture state. Generally, the pure state $|\Psi\rangle_3$ of Bobs EPR particle will reduce to a mixture state after a practical realization of unitary transformations, in the end Bob can not obtain a pure state of particle 3, and can not convert $|\Psi\rangle_3$ into the initial pure state $|\Psi\rangle$ which Alice sought to teleport. An unknown quantum state can thus not be teleported enough accurately because of the physical implementation of unitary transformations.

In the general, there exists deviation between the replica of unknown state in the hands of Bob after practical unitary transformation and the initial quantum state $|\Psi\rangle_1$ prepared by Alice. We can evaluate the above deviation by the difference between ρ_3 and the density matrix ρ_1 of pure state $|\Psi\rangle_1$, it is

$$\delta = \sqrt{\sum_{n,m} |(\rho_3)_{nm} - (\rho_1)_{nm}|^2}. \quad (4)$$

Considering the density matrix ρ_3 in the above expression (2), this deviation can be further written as

$$\begin{aligned} \delta^2 = & |C_0 a|^2 + |C_0 a|^2 |\langle E_0 | E_1 \rangle|^2 - |a|^2|^2 + |2C_0 C_1^* a b^* \langle E_1 | E_0 \rangle - a b^*|^2 \\ & + |2C_1 C_0^* b a^* \langle E_0 | E_1 \rangle - b a^*|^2 + |C_1 b|^2 + |C_1 b|^2 |\langle E_1 | E_0 \rangle|^2 - |b|^2|^2. \end{aligned} \quad (5)$$

We can see that the deviation is determined by the state vectors of environment. If we properly arrange the environment, hence the unitary transformation, and choose the state vector $|E_0\rangle = |E_1\rangle$, $C_0 = C_1 = \frac{1}{\sqrt{2}}$, then the deviation will vanish. In this special case, the state vector of combined particle 3-environment system is $|\Phi(t > t_1)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle \otimes (a|0\rangle_3 + b|1\rangle_3)$, there is no correlation between state vectors of particle 3 and environment at all, the quantum state of particle 3 still remain in a pure state after this special realization of unitary transformations. Certainly, that is a very special case and difficult to realize in an experiment. Generally speaking, we can not transfer an unknown state from one place to another enough accurately unless we choose the special physical implementation of unitary transformations.

Superficially, Bennett et al.'s treatment of quantum teleportation have not broken the quantum non-cloning theorem[9], because the initial unknown state has been destroyed after this teleportation. However, if we analyze this procedure carefully, we will find it surely break the quantum non-cloning theorem. The key point still lie in the physical realization of unitary transformations. In essence, quantum non-cloning theorem is similar to our above analysis. To clone an unknown quantum state $|\Psi\rangle = a|0\rangle + b|1\rangle$ of a spin-1/2 particle, we need the necessary apparatus and environment which are also summarily described as general environment. The outcome of the cloning procedure is

$a|A\rangle_0|0\rangle|0\rangle + b|A\rangle_1|1\rangle|1\rangle$, where $|A\rangle_i (i = 0, 1)$ are the quantum state vectors of apparatus, and the new state vectors $|0\rangle$ and $|1\rangle$ in the outcome are provided by the environment. There are correlation between the unknown state and the state vectors of environment. The general state vectors of environment $|A\rangle_i|i\rangle (i = 0, 1)$ just correspond to the state vectors $C_i|E_i\rangle (i = 0, 1)$ of environment in expression (1). So our above analysis is equivalent to quantum non-cloning theorem, it is the environment that leads to the impossibility for an unknown state being cloned accurately, the interaction of particle and environment will cause the decoherence of pure state of particle to a mixture state. To teleport an unknown quantum state enough accurately will eventually break the quantum non-cloning theorem because of the influence of environment.

In summary, we have shown that an unknown quantum state can not be teleported enough accurately from one place to another when we consider the practical physical implementation of unitary transformations except some special cases. The coupling of environment and particle will reduce the accuracy of converting procedure, it is another manifest of quantum non-cloning theorem.

Acknowledgments

This work is supported by the NNSF (Grant No. 10404037).

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